On the C*-envelope of approximately finite-dimensional operator algebras

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Abstract

The C*-envelope of the limit algebra (or limit space) of a contractive regular system of digraph algebras (or digraph spaces) is shown to be an approximately finite C*-algebra and the direct system for the C*-envelope is determined explicitly.

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A number of recent studies of non-self-adjoint operator algebras have been concerned with the Banach algebra direct limits of direct systems

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \quad . \ . \ . \ . \ A$$

in which the building block algebras are finite-dimensional digraph algebras. The most accessible of these arise when the algebra homomorphisms ϕ_k are regular in the sense that (partial) matrix unit systems can be chosen for $A_1, A_2, \ldots$ so that each ϕ_k maps matrix units to sums of matrix units. Under these conditions the maps ϕ_k need not be star-extendible or isometric and the C*-envelope of the limit algebra, in the sense of Hamana, need not be an approximately finite-dimensional C*-algebra. (See [6].)

In what follows we show that if the homomorphisms are regular and contractive then the C*-envelope $C^*_{env}(A)$ of the limit algebra A is an AF C*-algebra. Furthermore we identify explicitly a direct system of finite-dimensional C*-algebras for $C^*_{env}(A)$. In particular, this generalizes the result for the triangular compression limit algebras $\varinjlim(T_{n_k}, \phi_k)$ considered by Hopenwaser and Laurie [5] and answers the problem posed there. The relationship between A and $C^*_{env}(A)$ is rather subtle ; even after a natural telescoping of the given direct system to an essentially isometric system (Lemma 9) it need not be the case that $C^*_{env}(A)$ is an isometric direct limit of the algebras $C^*(\phi_k(A_k))$.

In the first section we consider only finite-dimensional matters. In particular the contractive regular morphisms are characterised as the regular bimodule maps of compression type. In the second section we obtain the main result and discuss a variety of examples.

1. Regular contractive morphisms

Let G be a finite directed graph with no multiple edges and with vertices labelled $1, 2, \ldots, n$. Let $\{e_{ij} : 1 \leq i, j \leq n\}$ be a matrix unit system for the full complex matrix algebra M_n. Define $A(G)$ to be the linear span of those matrix units e_{ij} for which (i, j) is an edge of G. If G is a reflexive digraph then we refer to $A(G)$ as a *digraph space*. If G is reflexive and transitive (as a binary relation) then we refer to $A(G)$ as a *digraph algebra*. In intrinsic terms a digraph algebra (alias finite-dimensional CSL algebra/poset algebra/ incidence algebra) is a subalgebra of M_n containing a maximal abelian self-adjoint subalgebra. Usually we consider the digraph algebras that are associated with the standard matrix unit system for M_n.

Definition 1. Let $A(G), A(H)$ be digraph spaces with associated (standard) matrix unit systems. Then a linear map $\phi : A(G) \to A(H)$ is said to be a *regular bimodule map*, with respect to the given matrix unit systems, if ϕ maps matrix units to orthogonal sums of matrix units and ϕ is a bimodule map with respect to the standard diagonal subalgebras. That is, if $C(G)$ and $C(H)$ are these diagonals, then $\phi(C(G)) \subseteq C(H)$ and

$$\phi(c_1 a c_2) = \phi(c_1)\phi(a)\phi(c_2)$$

for all c_1, c_2 in $C(G)$ and a in $A(G)$.

More generally a linear map between digraph spaces is said to be a *regular bimodule map* if matrix unit systems can be chosen so that the map is of the form above.

The terminology above should be compared with the following more general terminology (which is not needed in this paper). A map α between digraph

algebras is said to be regular if partial matrix unit systems can be chosen so that for the associated diagonals, say C and D respectively, α is a $C-D$ bimodule map and is regular in the sense that α maps the normaliser of C into the normaliser of D. In the case of contractive maps it can be shown that this notion coincides with that which is given in the definition above. That this notion is more general can be seen by considering the (Schur) automorphisms of the 4-cycle digraph algebra which leave the diagonal invariant.

If ϕ is as in Definition 1 then the image $\phi(e_{ij})$ of each matrix unit e_{ij} in $A(G)$ is, by assumption, a partial isometry which is an orthogonal sum of matrix units. Note also that the initial and final projections of $\phi(e_{ij})$ are dominated (perhaps properly) by the diagonal projections $\phi(e_{jj})$ and $\phi(e_{ii})$ respectively.

Definition 2. A diagonal projection Q in $A(G)$ is said to be $A(G)$-*irreducible* if $C^*(QA(G)Q) = QM_nQ = B(Q\mathbb{C}^n)$.

Plainly Q is $A(G)$-irreducible if the graph of the digraph algebra $QA(G)Q$ on $Q\mathbb{C}^n$ is connected as an undirected graph. Also, if Q is a diagonal projection of $A(G)$, then, by considering connected components of the graph of $QA(G)Q$, we can write $Q = \sum Q_i$, with Q_i diagonal and $A(G)$-irreducible, such that

$$QAQ = \sum_i \oplus Q_iAQ_i$$

Definition 3. The map $\phi : A(G) \to A(H)$ is a *(standard) elementary compression type map* associated with diagonal projections (Q^G, P^H) if

(i) Q^G and P^H are diagonal projections in $A(G)$ and $A(H)$, respectively, with

rank Q^G = rank $P^H = r$, and

(ii) Q^G is $A(G)$-irreducible, and

(iii) ϕ is a linear map of the form $\phi = \beta \circ \alpha$ where $\alpha : A(G) \to M_r$ is compression by Q^G and $\beta : M_r \to C^*(A(H))$ is a C*-algebra injection, with $\beta(I) = P^H$, which is a linear extension of a correspondence $e_{ij} \to f_{n_i n_j}$ of standard matrix units.

A (standard) elementary compression type map is simply the regular bimodule map associated with an identification of a connected full subgraph of G with an isomorphic subgraph of H. The minimal diagonal subprojections of P^H are associated with the vertices of the subgraph of H. Note that ϕ depends not only on the pair (Q^G, P^H) and the matrix unit system but also on the the particular identification β chosen for the pair. This dependence is often suppressed in the subsequent discussions.

A map $\phi : A(G) \to A(H)$ is a *(standard) compression type map* if it is a direct sum of elementary compression type maps.

More generally, ϕ is a *compression type map* if matrix unit systems can be chosen so that ϕ is of compression type with respect to these systems. Compression type maps feature in the linear topological aspects of limit algebras discussed in [8].

Lemma 4. *Let ϕ be the Schur product projection map which is defined on the digraph space of an m-cycle by deleting an entry corresponding to a proper edge. That is, consider the map*

$$A \in \begin{bmatrix} * & * & & & \\ & * & * & & \\ & & \ddots & & \\ & & & * & * \\ * & & & & * \end{bmatrix} \to \phi(A) \in \begin{bmatrix} * & * & & & \\ & * & * & & \\ & & \ddots & & \\ & & & * & * \\ 0 & & & & * \end{bmatrix}$$

where the entry a_{m1} of A is replaced by zero. Then the norm of the linear map ϕ dominates the quotient

$$\frac{\cos(\pi/(2m+1))}{\cos(\pi/2m)}.$$

In particular, this sparse triangular truncation map is not contractive.

Proof: Consider the following matrix A where, as usual, unspecified entries are zero :

$$A = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & & \\ & & & 1 & 1 \\ -1 & & & & 1 \end{bmatrix} \qquad (m \times m).$$

Let $D =$ diag$\{1, w, w^2, \ldots, w^{m-1}\}$ where $w = \exp(\pi i/m)$. Since $-\bar{w}^{m-1} = w$ we have

$$D^*AD = I + wS = \begin{bmatrix} 1 & w & & & \\ & 1 & w & & \\ & & \ddots & & \\ & & & 1 & w \\ w & & & & 1 \end{bmatrix}.$$

where S is the cyclic backward shift. Spectral theory then yields that

$$||I + wS|| = |1 + w| = 2\cos(\frac{\pi}{2m}).$$

On the other hand the truncate of A has norm equal to $2\cos(\frac{\pi}{2m+1})$. For details of this see, for example, Example 1.2.5 in [3]. $\square$

Theorem 5. *Let $\phi : A(G) \to A(H)$ be a regular bimodule map between digraph spaces. Then ϕ is contractive if and only if ϕ is of compression type.*

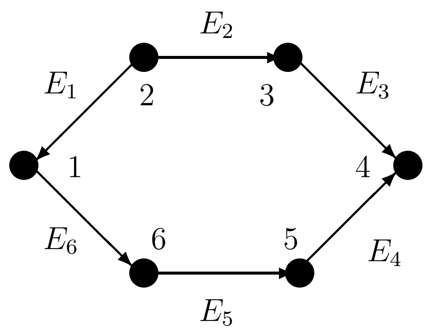
Proof: Let ϕ be a regular bimodule map with respect to the matrix unit systems $\{e_{ij}\}$ for $A(G)$ and $\{f_{kl}\}$ for $A(H)$. Choose f_{kk} in $A(H)$ such that f_{kk} is a summand of $\phi(e_{jj})$ for some j. Let $p_1 = f_{kk}$. Let $\mathcal{O} = \{p_1, p_2, \ldots, p_s\}$ be the set of distinct minimal diagonal projections in $A(H)$ which is the orbit of p_1 under $\{\phi(e_{ij}) : (i,j) \in E(G)\}$. This is the smallest set of projections which contains p_1 and is such that if p is in the set and $\phi(e_{ij})p \neq 0$ (or $p\phi(e_{ij}) \neq 0$) then $\phi(e_{ij})p(\phi(e_{ij}))^*$ (or $\phi(e_{ij})^* p\phi e_{ij})$ also belongs to the set. Each p_i is a summand of $\phi(e_{k_ik_i})$ for some k_i.

We claim that $k_i \neq k_j$ for $i \neq j$.

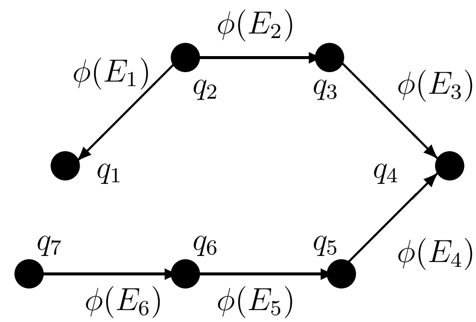
Suppose $k_i = k_j$ for some $i \neq j$. Then there is a subset $\{q_1, \ldots, q_{l+1}\}$ of the orbit with $q_1 = p_i$ and $q_{l+1} = p_j$ which corresponds to a cycle in G consisting of vertices $\{n_1, \ldots, n_l\}$ (with $n_1 = k_i$) and (directed) edges E_m connecting n_m and n_{m+1} for $m = 1, \ldots, l-1$, and edge E_l, connecting n_l and n_1 such that $\phi(E_m)$ maps q_m to q_{m+1} (or q_{m+1} to q_m depending on the direction of E_m), for $m = 1, \ldots, l$. Here we write E_m for the matrix unit determined by the edge E_m.

By relabelling the vertices of G appropriately we can assume that $\{n_1, \ldots, n_l\}$ is $\{1, \ldots, l\}$ and that the edge E_l runs from 1 to l. For example, the cycle might look like

In the graph G



In the algebra $A(H)$



Let

$$a = \sum_{m=1}^{l-1} E_m + \sum_{t \in T} e_{tt} - e_{l1}$$

where

$$T = \{m : E_m \text{ and } E_{m-1} \text{ run in the same direction, where } E_0 = E_l\}.$$

In the illustrated example,

$$a = e_{12} + e_{32} + e_{43} + e_{45} + e_{56} + e_{11} + e_{33} + e_{55} + e_{66} - e_{61}.$$

Then, after deleting rows of zeros and columns of zeros, a has a submatrix

of the form

$$A = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ -1 & & & & & 1 \end{bmatrix}.$$

Let $p = q_1 + \ldots + q_\ell$. Since $\phi(e_{\ell 1})$ maps $q_{\ell+1}$ into q_ℓ, $p\phi(e_{\ell 1})p = 0$ and hence $p\phi(a)p$ has the same norm as the associated matrix

$$\begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ 0 & & & & & 1 \end{bmatrix}.$$

By Lemma 3, $\| a \| < \| p\phi(a)p \|$ which contradicts the hypothesis that ϕ is contractive. Thus the claim is proven. That is, for each p_i in the orbit $\mathcal{O}$ with p_i a summand in $\phi(e_{k_i k_i})$, we have that $k_i \neq k_j$ if $p_i \neq p_j$.

Let $P = \sum p_i$, the sum taken over all p_i in $\mathcal{O}$. Then P is a diagonal projection in $A(H)$. Let $Q = \sum e_{k_i k_i}$, the sum taken over all k_i such that p_i is in $\mathcal{O}$. By the claim, Q is a diagonal projection in $A(G)$ with rank Q = rank P. Note that the graph of $QA(G)Q$ is connected so Q is $A(G)$-irreducible. Note also, from the definition of P, that $(I-P)\phi(A(G))P = 0$ and $P\phi(A(G))(I-P) = 0$.

Thus we can write $\phi = \gamma \oplus \phi'$ where $\gamma = P\phi$ and $\phi' = P^\perp \phi$.

There are two possibilities for γ. Either γ is an elementary compression type map (of multiplicity one) or γ fails to be injective. In the latter case there is a non-self-adjoint matrix unit, e_{ij} say, which is mapped to zero by γ (and ϕ) and which corresponds to an edge E of G, which is not one of the edges

$E_1, \dots E_l$, but which nevertheless has its two vertices in common with two of the vertices of $E_1, \dots E_l$. This means that E, together with some of the edges $E_1, \dots E_l$, form a cycle. The argument above applies and once again we obtain the contradiction $\|\gamma\| > 1$.

Induction completes the proof. $\square$

Corollary 6. *If $\phi : A(G) \to A(H)$ is a contractive regular bimodule map, then ϕ is completely contractive.*

Proof: For ϕ of compression type, $\phi^{(n)} : A(G) \otimes M_n \to A(H) \otimes M_n$ is also of compression type. $\square$

We now examine further properties of compression type maps. Let $B(\mathcal{M})$ denote all bounded operators on the Hilbert space $\mathcal{M}$ and write $\cong$ for C*-algebra isomorphism.

Proposition 7. *Let $A(G) \subseteq B(\mathbb{C}^n)$ and $A(H) \subseteq B(\mathbb{C}^m)$ be digraph algebras and let $\phi : A(G) \to A(H)$ be a compression type map. Thus $\phi = \sum \oplus \gamma_i$ where each γ_i is an elementary compression type map associated with the pair (Q_i^G, P_i^H) (and an implicit identification) with each Q_i^G being $A(G)$-irreducible. Then*

$$C^*(\phi(A(G))) \cong \sum_{k \in K} \oplus B(Q_k^G \, \mathbb{C}^n) \cong \sum_{k \in K} \oplus B(P_k^H \, \mathbb{C}^m)$$

where $\{Q_k^G : k \in K\}$ includes exactly one copy of each distinct Q_i^G.

Proof: We have $\gamma_i = \beta_i \circ \alpha_i$ where $\alpha_i : A(G) \to B(Q_i^G \mathbb{C}^n)$ is the compression map $a \to Q_i^G a Q_i^G$ and $\beta_i : B(Q_i^G \mathbb{C}^n) \to B(P_i^H \mathbb{C}^m)$ is a (standard) C*-algebra isomorphism. Define

$$\alpha : A(G) \to \sum_i \oplus B(Q_i^G \mathbb{C}^n)$$

and

$$\beta : \sum \oplus B(Q_i^G \mathbb{C}^n) \to \sum \oplus B(P_i^H \mathbb{C}^m)$$

by $\alpha(a) = \sum \oplus\ \alpha_i(a)$ and $\beta(\sum \oplus b_i) = \sum \oplus \beta_i(b_i)$. Thus $C^*(\phi(A(G)))$ is isomorphic to $C^*(\alpha(A(G)))$ which in turn is isomorphic to $C^*(\hat{\alpha}(A(G)))$ where $\hat{\alpha}(a) = \sum_{i \in K} \oplus \alpha_i(a)$.

Thus $C^*(\phi(A(G))$ is identified with a C*-subalgebra, E say, of

$$F = \sum_{k \in K} \oplus B(Q_k^G \mathbb{C}^n).$$

The compression of E to each summand of F is equal to the summand, so it remains to show only that no summand of E appears with multiplicity in the summands of F. This is elementary. For example if Q_k^G and Q_l^G are distinct, with $k, l \in K$, and $\operatorname{rank}(Q_k^G) = \operatorname{rank}(Q_l^G)$ then consider a self-adjoint matrix unit e of $A(G)$ with $Q_k^G e = e$ and $Q_l^G e = 0$. Then $\phi(e)$ differs in the summands for Q_k^G and Q_l^G being nonzero in one summand and zero in the other, as desired. $\square$

Proposition 8. *Let $\gamma : A(G) \to A(H)$ be an elementary compression type map with projection pair (Q^G, P^H) and let $\eta : A(H) \to A(F)$ be an elementary compression type map with projection pair (Q^H, P^F). Then $\eta \circ \gamma : A(G) \to A(F)$ is of compression type with $\eta \circ \gamma = \sum \oplus \delta_i$ where δ_i is an elementary compression type map with projection pair (q_i^G, p_i^F) where the q_i^G are orthogonal subprojections of Q^G and the p_i^F are orthogonal subprojections of P^F.*

Proof: Suppose $P^H = \sum_{k\in K} f_{kk}$ so that $P^H Q^H = \sum_{j\in J} f_{jj}$ for some subset J of K. Each $f_{jj}, j \in J$, corresponds via γ with a minimal diagonal projection $e_{k_j k_j}$ which is a summand of Q^G. Let $q^G = \sum_{j\in J} e_{k_j k_j}$ and write $q^G = \sum q_i^G$ as a direct sum of $A(G)$-irreducible projections. Each q_i^G corresponds under γ to a diagonal subprojection, p_i^H, of $P^H Q^H$, which in turn corresponds under η to a diagonal subprojection p_i^F of P^F. Thus $\eta \circ \gamma = \sum \delta_i$ where δ_i is an elementary compression type map associated with (q_i^G, p_i^F). $\square$

2. Regular direct systems of digraph spaces

We turn our attention now to systems

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \quad \dots .$$

where each A_i is a digraph space $A(G_i)$ for some digraph G_i and each ϕ_i is a contractive regular bimodule map. We refer to such a system as a *contractive regular direct system of digraph spaces.*

Define

$$A_\infty^0 = \{(a_k) : a_k \in A_k, \phi_k(a_k) = a_{k+1} \text{ for all large } k\}$$

and let A_∞ be the set of equivalence classes of eventually equal sequences. Let $\phi_{k,\infty}$ denote the natural map from A_k into A_∞ and define the seminorm $\|\ \|$ on A_∞ by $\| \phi_{k,\infty}(a) \| = \limsup_l \| \phi_\ell \circ \ldots \circ \phi_k(a) \|$ for $a \in A_k$. Then the quotient of A_∞ by the subspace of elements with zero seminorm becomes a normed space. The direct limit, A, of the system is defined to be the completion of this normed space. In fact the Banach space is matricially normed in the obvious way and the maps $\phi_{k,\infty}$ are completely contractive.

If the A_i's are algebras then A_∞ has a natural algebra structure, the induced (operator) norm is an algebra norm, and the direct limit is a Banach algebra. In this case (see [6]) A is completely isomorphic to a Hilbert space

operator algebra. If the A_i's are C*-algebras and the ϕ_i's are contractive star homomorphisms (not necessarily injective), then A_∞ inherits a natural star algebra structure, the norm defined above is a C*-norm, and the direct limit is a C*-algebra.

Lemma 9. *The system above can be replaced by a subsystem*

$$\mathcal{A}_1 \xrightarrow{\alpha_1} \mathcal{A}_2 \xrightarrow{\alpha_2} \quad .\ .\ .\ .$$

such that

(i) each α_i is of compression type,

(ii) the set of $\mathcal{A}_k$-irreducible projections associated with $\alpha_l \circ \ldots \circ \alpha_k$ for each $\ell > k$ is the same as the set of $\mathcal{A}_k$-irreducible projections associated with α_k,

(iii) the restriction of each map α_{k+1} to $\alpha_k(\mathcal{A}_k)$ is isometric.

Proof: A composition $\beta \circ \alpha$ of compression type maps is of compression type. Furthermore, the compression projections for the compositions are subprojections of the compression projections for α (by Proposition 8). In view of this we may choose k so that the compression projections for the composition

$$\phi_\ell \circ \ldots \circ \phi_1$$

is constant for all $l > k$. It is now clear how to similarly choose the maps α_k to satisfy the conditions (i) and (ii). Property (iii) now follows. $\square$

Theorem 10. *Let $\{A_k, \phi_k\}$ be a contractive regular direct system of digraph spaces and let $\{\mathcal{A}_k, \alpha_k\}$ be an essentially isometric subsystem with the prop-*

erties of Lemma 9. Then there exists isometric star homomorphisms ψ_k such that the diagrams

$$\begin{array}{ccc} \alpha_k(\mathcal{A}_k) & \xrightarrow{\alpha_{k+1}} & \alpha_{k+1}(\mathcal{A}_{k+1}) \\ i_k \downarrow & & \downarrow i_{k+1} \\ C^*(\alpha_k(\mathcal{A}_k)) & \xrightarrow{\psi_k} & C^*(\alpha_{k+1}(\mathcal{A}_{k+1})) \end{array}$$

commute.

Proof: By Theorem 5 we may assume that the maps are of compression type. Let $\mathcal{A}_k \subseteq M_{m_k}$ be the natural inclusion. Let $\alpha_k = \gamma = \sum_{r=1}^{t} \oplus \gamma_r$ where each γ_r is an elementary compression type map associated with diagonal projections (Q_r^k, P_r^{k+1}), and let $\alpha_{k+1} = \sum \oplus \eta_s$ where η_s is associated with diagonal projections (Q_s^{k+1}, P_s^{k+2}). Let I be a subset of indices such that $\{Q_i^k : i \in I\}$ contains exactly one copy of each Q_r^k that appears in the description of α_k. Let J be a subset of indices such that $\{Q_j^{k+1} : j \in J\}$ contains exactly one copy of each Q_s^{k+1} that appears in the description of α_{k+1}.

By Proposition 7, we have the isomorphisms

$$C^*(\alpha_k(\mathcal{A}_k)) \cong \sum_{i \in I} \oplus B(Q_i^k \mathbb{C}^{m_k})$$

and

$$C^*(\alpha_{k+1}(\mathcal{A}_{k+1})) \cong \sum_{j \in J} \oplus B(Q_j^{k+1} \mathbb{C}^{m_{k+1}}) \cong \sum_{j \in J} \oplus B(P_j^{k+2} \mathbb{C}^{m_{k+2}}).$$

We shall define $\psi_k : C^*(\alpha_k(\mathcal{A}_k)) \to C^*(\alpha_{k+1}(\mathcal{A}_{k+1}))$ in accordance with the multiplicity and the identification of the embedding of each Q_i^k-summand into each P_j^{k+2}-summand.

By Proposition 8, $\eta_j \circ \gamma = \sum \oplus \delta_\ell$ where δ_ℓ is an elementary compression type map associated with projections (q_ℓ, p_ℓ) where $\{p_\ell\}$ consists of mutually orthogonal subprojections of P_j^{k+2}. By the hypotheses (that of Lemma 9 (ii) in this case), each q_ℓ belongs to $\{Q_i^k : i \in I\}$. Note that if $Q_j^{k+1} = Q_m^{k+1}$ then the $\{q_\ell\}$ associated with $\eta_j \circ \gamma$ and the $\{q_\ell\}$ associated with $\eta_m \circ \gamma$ are identical (counting multiple copies).

We are now ready to define ψ_k. For $i \in I$ and $j \in J$, let n_{ij} be the number of copies of Q_i^k that occur in the set of $\{q_\ell\}$ in the description above of $\eta_j \circ \gamma$. Then there is a natural embedding of multiplicity n_{ij} of the Q_i^k-summand of $C^*(\alpha_k(\mathcal{A}_k))$ into the P_j^{k+2}-summand of $C^*(\alpha_{k+1}(\mathcal{A}_{k+1}))$. This map is just the star extension of the restriction of $\eta_j \circ \gamma$.

By assumption on the maps α_k, given $i \in I$, there exists $j \in J$ such that $n_{ij} \neq 0$. (Otherwise $\alpha_{k+1} \circ \gamma_i = 0$ implying that Q_i^k does not appear in the set of projections associated with $\alpha_{k+1} \circ \alpha_k$). Thus ψ_k is isometric.

To see that the diagram commutes one can argue as follows. Consider an element $b = \alpha_k(a)$ with $a \in \mathcal{A}_k$. Then b splits as a direct sum $b_1 + \ldots + b_t$ with $b_r = P_r^{k+1} b_r P_r^{k+1}$ where $P_1^{k+1}, \ldots, P_t^{k+1}$ is the enumeration of all the projections in the definition of α_k. Furthermore, we can group the summands according to the equivalence relation $r \sim s$ on indices with $Q_r^{k+1} = Q_s^{k+1}$ to obtain

$$b = \sum_{i \in I} \oplus (\sum_{r \in i} \oplus b_r).$$

Here we view the indices in I also as the equivalence classes. Each summand b_r for $r \in i$ is the copy $\gamma_r(Q_r^k a Q_r^k)$ of $Q_i^k a Q_i^k$, and the map i_k is the natural map which identifies these copies. That is, $i_k(b) = \sum_{i \in I} \oplus b_i$.

Similarly, each element d of $\alpha_{k+1}(\mathcal{A}_{k+1})$ has a direct sum representation

$$d = \sum_{j \in J} \oplus (\sum_{s \in j} \oplus d_s).$$

and i_{k+1} is the map with $i_{k+1}(d) = \sum_{j \in J} \oplus d_j$ which identifies multiple copies as before.

Consider the representation of $\alpha_{k+1}(b)$ in the form $\sum_{j \in J} \oplus (\sum_{s \in j} \oplus d_s)$. Then each summand d_s, coming from $(\eta_s \circ \gamma)(b)$, is itself a direct sum of copies of b_i in the expression for d_j. The implication of this is that there is a commuting diagram

$$\begin{tikzcd}
\alpha_k(\mathcal{A}_k) \arrow[r, "\alpha_{k+1}"] \arrow[d, "i_k"'] & \alpha_{k+1}(\mathcal{A}_{k+1}) \arrow[d, "i_{k+1}"'] \\
\{\sum_{i \in I} \oplus b_i\} \arrow[r, "\hat{\psi}_{k+1}"] & \{\sum_{j \in J} \oplus d_j\}
\end{tikzcd}$$

The map ψ_{k+1} is simply the star-extension of the map $\hat{\psi}_{k+1}$ and so the diagram in the statement of the theorem commutes. □

Corollary 11. *The direct limit of a contractive regular system of digraph spaces is completely isometric to a subspace of an AF C*-algebra.*

Proof: In the notation above consider the following commuting diagram for such a direct limit A.

$$
\begin{tikzcd}
\alpha_1(\mathcal{A}_1) \arrow[r, "\alpha_2"] \arrow[d, "i_1"'] & \alpha_2(\mathcal{A}_2) \arrow[r] \arrow[d, "i_2"'] & \cdots & A \arrow[d, "i"'] \\
C^*(\alpha_1(\mathcal{A}_1)) \arrow[r, "\psi_2"] & C^*(\alpha_2(\mathcal{A}_2)) \arrow[r] & \cdots & B
\end{tikzcd}
$$

Since the inclusion maps i_k are completely isometric we conclude that the induced map $i : A \to B$ is a complete isometry. $\square$

Proposition 12. *Given the systems*

$$
\begin{tikzcd}
\alpha_1(A_1) \arrow[r, "\alpha_2"] \arrow[d, "i_1"'] & \alpha_2(A_2) \arrow[r] \arrow[d, "i_2"'] & \cdots & A \arrow[d, "i"'] \\
C^*(\alpha_1(A_1)) \arrow[r, "\psi_2"] & C^*(\alpha_2(A_2)) \arrow[r] & \cdots & B
\end{tikzcd}
$$

as in Theorem 10, define $f_k = \alpha_{k,\infty}(I_k)$ where I_k is the identity in $\mathcal{A}_k$. Then

(a) the sequence $e_k = i(f_k)$ forms a norm approximate identity for B

(b) the following statements are equivalent

i) ψ_i is unital for all large i.

ii) B has a unit e.

iii) e_k converges in norm to an element e in B.

iv) f_k converges in norm to an element g in A.

Under any of the assumptions in (b) we have that $i(g) = e$.

Proof: The statements follow directly once it is observed that $i_k(\alpha_k(I_k))$ is the identity in B_k and that $i \circ \alpha_{k+1,\infty} = \psi_{k,\infty} \circ i_k$. $\square$

Definition 13. Let

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \quad . \; . \; . \; . \; A$$

be a direct system of digraph spaces where each ϕ_i is a contractive regular bimodule map. This system is said to be *essentially unital* if the telescoped system given by Lemma 9 satisfies any of the equivalent properties in (b) of Proposition 12.

Before going further we provide a brief review of the idea of a C*-envelope.

The appropriate setting, considered by Hamana ([4]), is the category of unital operator spaces (that is, self-adjoint unital subspaces of C*-algebras), together with unital complete order injections. If A is an operator space, then a C*-*extension* of A is a C*-algebra B together with a unital complete order injection ρ of A into B such that $C^*(\rho(A)) = B$. A C*-extension B is a C*-*envelope* of A provided that, given any operator system, C, and any unital completely positive map $\tau : B \to C$, τ is a complete order injection whenever $\tau \circ \rho$ is. Hamana proves the existence and uniqueness (up to a suitable notion of equivalence) of C*-envelopes. Furthermore , he shows that the C*-envelope of A is a minimal C*-extension in the family of all C*-extensions of A.

After proving the existence of C*-envelopes, Hamana then uses this to prove the existence of a Silov boundary for A, a theme first developed by Arveson [1]. The Silov boundary is a generalization to operator spaces, of the usual notion of Silov boundary from function spaces.

Let B be a C*-algebra and let A be a unital sub-operator-space such that $B = C^*(A)$. An ideal J in B is called a *boundary ideal* for A if the canonical quotient map $B \to B/J$ is completely isometric on A. A boundary ideal exists which contains every other boundary ideal and this maximal boundary ideal is called the *Silov boundary* for A. Hamana shows that if B is a C*-extension for A, then the C*-envelope for A is isomorphic to B/J, where J is the Silov boundary for A. It is this form of the definition of the C*-envelope that will be used below.

If A is merely a unital subspace of a C*-algebra, rather than an operator space, then we define the C*-envelope of A to be the C*-envelope of the operator space $A + A^*$.

Proposition 14. *Let A be the direct limit of an essentially unital contractive regular direct system of digraph spaces. Then A is completely isometrically isomorphic to a unital subspace of an AF C*-algebra. Furthermore the C*-envelope of A is an AF C*-algebra.*

Proof: Consider the telescoped system and the associated commuting diagram

$$\begin{array}{ccccccc}
\alpha_1(\mathcal{A}_1) & \xrightarrow{\alpha_2} & \alpha_2(\mathcal{A}_2) & \longrightarrow & \cdots & & A \\
i_1 \big\downarrow & & i_2 \big\downarrow & & \cdots & & i \big\downarrow \\
B_1 & \xrightarrow{\psi_2} & B_2 & \longrightarrow & \cdots & & B
\end{array}$$

where $B_i = C^*(\alpha_i(\mathcal{A}_i))$ is a finite-dimensional C*-algebra, each ψ_i is an isometric star homomorophism and i is a complete isometry. Since the telescoped system is unital, we see from the definition of ψ_i that each ψ_i is unital. Thus, by Proposition 12 there exists $g \in A$ such that $i(g) = e$, the unit in B and the first assertion of the theorem follows.

Let A also denote the image of A in B. Then $\tilde{A} = \overline{A + A^*}$ is an operator space and, since i is a complete isometry on A it extends to a unital complete order injection $i : \tilde{A} \to B$. Thus B is a C*-extension of $\tilde{A}$ and $C^*_{env}(A)$ is (completely isometric to) the quotient B/J where J is the Silov boundary of $\tilde{A}$ in B. $\square$

The direct system for $C^*_{env}(A)$

We shall now identify the direct system for $C^*_{env}(A)$. This requires an identification of the Silov ideal J above, which, as we see in Example 16 below, may be nonzero. This example shows that we cannot expect $C^*_{env}(A)$ to be an isometric limit of the C*-algebras $C^*(\alpha_k(A_k))$ of the isometric telescoped system for A. Nevertheless, because of the nature of ideals in direct systems it follows that the C*-envelope is a direct limit of these building blocks with respect to not necessarily injective embeddings.

View B_k as the isometric image of B_k in B and let $J_k = J \bigcap B_k$. Then

we have $J = \overline{\bigcup J_k}$ and isometric *-homomorphisms $\tilde{\psi}_i$ exist such that the following diagram commutes. (See Bratteli [2] for example.)

$$\begin{tikzcd}
B_1 \arrow[r, "\psi_1"] \arrow[d, "\pi_1"'] & B_2 \arrow[r] \arrow[d, "\pi_2"'] & \cdots\cdots & B \arrow[d, "\pi"'] \\
B_1/J_1 \arrow[r, "\tilde{\psi}_1"] & B_2/J_2 \arrow[r] & \cdots\cdots & B/J
\end{tikzcd}$$

Since J is a boundary ideal of $\tilde{A}$ in B, the map $\pi : B \to B/J$ is completely isometric on $\tilde{A}$. Hence $\pi_k : B_k \to B_k/J_k$ is completely isometric on $i_k(\alpha_k(A_k))$. We thus have the commuting system

$$\begin{tikzcd}
\alpha_1(A_1) \arrow[r, "\alpha_2"] \arrow[d, "\tilde{i}_1"'] & \alpha_2(A_2) \arrow[r] \arrow[d, "\tilde{i}_2"'] & \cdots\cdots & A \arrow[d, "\tilde{i}"'] \\
B_1/J_1 \arrow[r, "\tilde{\psi}_2"] & B_2/J_2 \arrow[r] & \cdots\cdots & B/J = C^*_{env}(A)
\end{tikzcd}$$

where $\tilde{i_k} = \pi_k \circ i_k$ is a complete isometry and $\tilde{i} = \pi \circ i$ is a complete isometry.

Furthermore, as Ken Davidson has noted, the identifications of Theorem 10 allow the Silov boundary J to be identified in the following specific intrinsic manner.

Consider the summands of $C^*(\alpha_k(A_k))$ which are *not maximal* in the sense that they correspond to proper compressions of larger summands of $C^*(\alpha_k(A_k))$. Otherwise refer to a summand as *maximal.* If a summand of $C^*(\alpha_k(A_k))$ is never mapped into a maximal summand of $C^*(\alpha_j(A_j))$ for any $j > k$ then (the image of) this summand is clearly contained in the Silov ideal. (In fact

such a summand generates a boundary ideal.) Moreover, the Silov boundary J is precisely the ideal, K say, that is generated by all such summands.

To see this suppose, by way of contradiction, that J contains K strictly. Then there is a summand, S say, of $C^*(\alpha_k(A_k))$ for some k, which is contained in J and which has (partial) embeddings into maximal summands $M_{n_j} \subseteq C^*(\alpha_{n_j}(A_{n_j}))$, for some increasing sequence n_j. Let $a \in \alpha_k(A_k)$ be an element such that $\|i_k(a)+S\| < \|a\|$. For example, pick $a = \alpha_k(b)$ where b is supported by a single compression projection (for the domain of α_k) corresponding to S and all proper subcompressions of b have strictly smaller norm. Plainly $\|\alpha_{k+1,\infty}(\tilde{i}(a)) + J\| \leq \|i_k(a) + S\| < \|a\|$. But the existence of embeddings into maximal summands implies that $\|\alpha_{k+1,\infty}(\tilde{i}(a))\| = \|a\|$. This is contrary to the fact that J is a boundary ideal and so the assertion follows.

In summary then we have the following theorem.

Theorem 15. Let A be the direct limit of an essentially unital contractive regular direct system of digraph spaces A_k. Then

(i) A is completely isometrically isomorphic to a unital subspace of an AF C^*-algebra.

(ii) $C^*_{env}(A)$ is an AF C^*-algebra.

(iii) If $B = \varinjlim C^*(\alpha_i(\mathcal{A}_i))$ is the C^*-algebra of the telescoped system for A (as in Theorem 10) then $C^*_{env}(A) = B/J$ where J is the ideal generated by those summands of $C^*(\alpha_i(\mathcal{A}_i))$, for $i = 1, 2, \ldots$, which have no partial embedding into a maximal summand of $C^*(\alpha_j(\mathcal{A}_j))$ for all $j > i$.

Remark. Note that if $\phi : A(G) \to A(H)$ is a contractive regular bimodule map, then since ϕ is of compression type, ϕ can be extended to $\tilde{\phi} : A(G) + A(G)^* \to A(H) + A(H)^*$ where $\tilde{\phi}$ is of compression type associated with the

same projections as ϕ. It follows that the commuting diagram above can be interpolated to yield

$$
\begin{tikzcd}
\alpha_1(A_1) \arrow[r, "\alpha_2"] \arrow[d] & \alpha_2(A_2) \arrow[r] \arrow[d] & {} \arrow[r] & A \arrow[d] \\
\alpha_1(A_1 + A_1^*) \arrow[r, "\tilde{\alpha}_2"] \arrow[d] & \alpha_2(A_2 + A_2^*) \arrow[r] \arrow[d] & \cdots\cdots \arrow[r] & \tilde{A} \arrow[d] \\
C^*(\alpha_1(A_1))/J_1 \arrow[r, "\tilde{\psi}_1"] & C^*(\alpha_2(A_2))/J_2 \arrow[r] & {} \arrow[r] & B/J = C^*_{env}(A)
\end{tikzcd}
$$

Consequently the operator space $\tilde{A} \equiv \overline{i(A) + i(A)^*}$ is completely isometric to the limit of the middle system.

Example 16 The following example illustates how the ideal J of Theorem 15 (iii) may be proper, even when the system $\{C^*(\alpha_i(\mathcal{A}_i)), \psi_i\}$ is isometric.

Consider

$$
T_3 \xrightarrow{\alpha_1} T_5 \xrightarrow{\alpha_2} \quad . \ . \ . \ . \ A
$$

where T_n is the algebra of $n \times n$ upper triangular matrices and where $\alpha_i(a) = a_{22} \oplus pap \oplus pap$ where a_{22} is the compression of a to the second minimal diagonal projection and p is compression to the last $n-1$ minimal diagonal projections.

For example,

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{bmatrix} \to \begin{bmatrix} a_{22} & & \\ & C & \\ & & C \end{bmatrix} = \alpha(T_3) \ \textit{where} \ C = \begin{bmatrix} a_{22} & a_{23} \\ & a_{33} \end{bmatrix},
$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{22} & a_{23} & a_{24} & a_{25} \\ & & a_{33} & a_{34} & a_{35} \\ & & & a_{44} & a_{45} \\ & & & & a_{55} \end{bmatrix} \to \begin{bmatrix} a_{22} & & \\ & D & \\ & & D \end{bmatrix} = \alpha_2(T_5)$$

where $D = \begin{bmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ & a_{33} & a_{34} & a_{35} \\ & & a_{43} & a_{45} \\ & & & a_{55} \end{bmatrix}$, and thus

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{bmatrix} \to \begin{bmatrix} a_{22} & & & & \\ & C & & & \\ & & C & & \\ & & & C & \\ & & & & C \end{bmatrix} = (\alpha_2 \circ \alpha_1)(T_3)$$

where C is as above.

Note that the system satisfies the properties of Lemma 9. Let Q_1 be the second minimal diagonal projection in T_3 and let Q_2 be the sum of the last two minimal diagonal projections in T_3. Let P_1 be the first minimal diagonal projection in T_9, P_2 the sum of the second through the fifth minimal diagonal projections in T_9, and P_3 the sum of the sixth through the ninth minimal diagonal projections in T_9. We can view $\alpha_2 \circ \alpha_1$ as embedding $B(Q_1 \mathbb{C}^3)$ into $B(P_1 \mathbb{C}^9)$ with multiplicity 1 and embedding $B(Q_2 \mathbb{C}^3)$ into $B(P_2 \mathbb{C}^9)$ with multiplicity 2 (as well as $B(Q_2 \mathbb{C}^3)$ into $B(P_3 \mathbb{C}^9)$ with multiplicity 2).

Letting M_i denote the $i \times i$ matrices, we have

$$C^*(\alpha_1(T_3)) \cong B(Q_1 \mathbb{C}^3) \oplus B(Q_2 \mathbb{C}^3) \cong M_1 \oplus M_2$$

and

$$C^*(\alpha_2(T_5)) \cong B(P_1\mathbb{C}^9) \oplus B(P_2\mathbb{C}^9) \cong M_1 \oplus M_4.$$

The embedding $\psi_2 : C^*(\alpha_1(T_3)) \to C^*(\alpha_2(T_5))$ induced by $\alpha_2 \circ \alpha_1$ is represented by the Bratteli diagram

$$\begin{array}{ccc} M_1 & \oplus & M_2 \\ | & & \| \\ M_1 & \oplus & M_4 \end{array} .$$

This Bratteli diagram format continues for

$$B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} \quad .\ .\ .\ .\ B$$

where $B_i \cong M_1 \oplus M_{2^i}$.

Let J be the ideal in $B = \lim\limits_{\to}(B_i, \psi_i)$ that corresponds to the subdiagram of embeddings of $M_1 \oplus 0$ into $M_1 \oplus 0$. Then $\pi : B \to B/J$ is completely isometric on A (deleting the corner entry a_{22} does not affect the norm of any of the images) and so J is a boundary ideal for A. and B is not the C*-envelope of A.

In fact, as we see from the general discussion below, we can make the identifications $C^*_{env}(A) \cong B/J = \lim\limits_{\to}(B_i/J_i, \tilde{\psi_i})$ where $J_i = M_1 \oplus 0 \subseteq B_i$ and so $C^*_{env}(A)$ is the UHF(2^∞) Glimm algebra.

Final remarks

We conclude with comments on various contractive regular systems.

Consider the system

$$M_n \xrightarrow{\phi_1} M_{n+k_1} \xrightarrow{\phi_2} \quad .\ .\ .\ .\ B$$

with embeddings ϕ_j such that

$$\phi_j(a) = a \oplus a_{p_j p_j} I_{k_j}$$

where $a_{p_j p_j}$ is the last diagonal entry of the matrix a. Here k_j is a sequence of positive integers and $p_j = n + k_1 + \ldots + k_{j-1}$. These embeddings restrict to algebra injections of the upper triangular matrix subalgebras, giving a triangular limit algebra, the limit algebra A of Example B in [5].

In a sense this example is not properly of compression type because the limit space B can be viewed as a limit of a subsystem for which the embeddings restrict to star algebra homomorphisms. Indeed, let $B_j \subseteq M_{p_j}$ be the block diagonal subspace $M_{p_j - 1} \oplus \mathbb{C}$. Then the maps $\phi_j : B_{j-1} \to B_j$ are C*-algebra embeddings, and, furthermore, the subsystem $\{B_{j-1}, \phi_j\}$ has the same limit, B. In particular the triangular operator algebra A is a regular subalgebra of an AF C*-algebra in the usual sense ([7]). Here $B = C^*(A)$ is the C*-envelope of A and, furthermore, the masa $A \cap A^*$ in A is a masa in $C^*_{env}(A)$. Similar remarks apply to the Examples D, E, F of [5].

On the other hand (as noted in [5]) the system $T_n \to T_{n+1} \to T_{n+2} \ldots$ with algebra homomorphisms $a \to a \oplus a_{ii}$, where $1 < i < n$ is fixed, is properly of compression type; the containing system $M_n \to M_{n+1} \to M_{n+2} \ldots$ does not have an algebra subsystem (in the sense above) with the same limit. Indeed in this case the image of the masa $A \cap A^*$ in A under the inclusion $A \to C^*_{env}(A)$ is not maximal abelian. For this example one readily identifies the C*-envelope as the AF C*-algebra with Bratteli diagram

$$\begin{tikzcd}[arrows=dash]
\mathrm{n} \arrow[d] & 1 \arrow[d] \arrow[dl] \\
\mathrm{n+1} \arrow[d] & 1 \arrow[d] \arrow[dl] \\
\mathrm{n+2} & 1
\end{tikzcd}$$

In principle, the enveloping C*-algebra of a given (proper) compression type system of digraph spaces can be identified by explicating the Bratteli diagram from the process described in the Lemma 9, Theorem 14 and the related discussions. Nevertheless, combinatorial counting arguments may be necessary for this which make this difficult in practice, as is the case, for example, with the system $T_2 \to T_3 \to T_6 \to \ldots$, with unital embedding homomorphisms that have exactly one copy of every proper interval compression.

It should be apparent from our discussions that the major aspect determining the C*-envelope is the nature and the number of the compressions appearing in the morphisms of the given direct system. The identity of the building blocks themselves plays a minor role. In fact, for any given AF C*-algebra B one can construct a regular contractive system $T_{2^{N_1}} \to T_{2^{N_2}} \to \ldots$, with algebra homomorphisms, such that the triangular limit algebra has C*-envelope equal to B.

To see this let $\phi' : M_{n_1} \oplus \ldots \oplus M_{n_p} \to M_{m_1} \oplus \ldots \oplus M_{m_q}$ be any standard C*-algebra homomorphism and let $\theta_1 : A_1 \to A_2$ be the restriction of ϕ' to the upper triangular subalgebras. Choose N_1, N_2 with $2^{N_1} \geq n_1 + \ldots + n_p$, $2^{N_2} \geq m_1 + \ldots + m_q$ and let $\kappa_i : T_{2^{N_i}} \to A_i$ be the natural block diagonal compression maps. Then the map $\alpha_1 = \kappa_2^{-1} \circ \theta_1 \circ \kappa_1$ is a contractive regular algebra homomorphism from A_1 to A_2. Iterating this construction one can express $\varinjlim(A_k, \theta_k)$ as $\varinjlim(T_{2^{N_k}}, \alpha_k)$ and the assertion follows.

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